Constructive influence of noise flatness and friction on the resonant behavior of a harmonic oscillator with fluctuating frequency

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The influences of noise flatness and friction coefficient on the long-time behavior of the first two moments and the correlation function for the output signal of a harmonic oscillator with fluctuating frequency subjected to an external periodic force are considered. The colored fluctuations of the oscillator frequency are modeled as a trichotomous noise. The study is a follow up of the previous investigation of a stochastic oscillator [Phys. Rev. E **78**, 031120 (2008)], where the connection between the occurrence of energetic instability and stochastic multiresonance is established. Here we report some unexpected results not considered in the previous work. Notably, we have found a nonmonotonic dependence of several stochastic resonance characteristics such as spectral amplification, variance of the output signal, and signal-to-noise ratio on the friction coefficient and on the noise flatness. In particular, in certain parameter regions spectral amplification exhibits a resonancelike enhancement at intermediate values of the friction coefficient.

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In recent years increasing attention has been focused on the constructive role of noise in nature: the influence of noise is not restricted to destructive and thermodynamic effects but can have unexpected ordered outcomes [1]. Examples include the ratchet effect [2], noise-induced multistability as well as first-order phase transitions in some complex systems [3], noise-induced Hopf bifurcations [4], and stochastic resonance (SR) [5], to name a few.

Since Benzi *et al.* [6] originally discovered the phenomenon of SR, whereby the effect of a small periodic force can be amplified by noise in a nonlinear system, there have been considerable developments in the process of exploring the essence of SR. Compared with the previous concept, in the last decade two generalizations can be discerned. First, it is generally acknowledged that in a wide sense, SR means a nonmonotonic dependence of the output signal or some function thereof [moments, autocorrelation functions, and signal-to-noise ratio (SNR)] on the noise parameters [7]. Second, the previous concept of SR was confined to systems where all three ingredients (nonlinearity, periodic, and random forces) are necessary for the onset of SR. However, recent investigations show that SR may appear without a periodic force [8] and in linear systems with multiplicative noise [7,9]. Since the phenomenon is very broad, there is still no general agreement about the precise conditions for its occurrence, its meaning, or even its defining characteristic signature. In the course of time a considerable variety of different quantifiers for SR have been introduced, leading to different quantitative conclusions regarding the occurrence of SR in a physical system (see [10]). For example, if we focus on a bona fide SR—an optimal signal enhancement phenomenon upon variation in the driving frequency—the most common quantifiers of SR are the signal-to-noise ratio, the spectral power amplification, and the hysteresis loops area. Often, a nonmonotonic resonance-type behavior of one of those three quantities is considered as the defining characteristic signature of SR. However, in many cases, it has been found that some of the three quantities behave monotonically and some nonmonotonically for same physical system [10]. Thus, in a general case different quantifiers of SR should be considered as complementary characteristics of SR. It should be noted that in a wide sense SR is related with the phenomena of stochastic parametric resonance [11,12] and stochastic resonant damping [1]. In those cases the prominent quantifier of SR is the variance of the output signal. However, while stochastic resonant damping is associated with the minimization of the system output variance, stochastic parametric resonance is considered either with the maximization of the output variance or with the phenomenon of energetic instability, which manifests itself in an unlimited increase in the second-order moments of the output with time.

Theoretical investigations [13,14] indicate that noise-induced nonequilibrium effects are sensitive to noise flatness, which is defined as the ratio of the fourth moment to the square of the second moment of the noise process. For example, in correlation ratchets (Brownian motors) the direction of the mean particle velocity crucially depends on the values of noise flatness [13,14]. The study of such systems has been motivated, in part, by recent advances in experimental study of motor proteins, i.e., proteins that convert the energy of adenosine triphosphate (ATP) hydrolysis into motion along a biopolymer. It is speculated that a multiplicative noise with several discrete states is likely to be the operating principle for motor proteins [15]. The stochastic binding of ATP, the subsequent hydrolysis, and the release of adenosine diphosphate (ADP) cause fluctuations of the distribution of charges in the motor protein and thus the energy profile that is “left” on the periodic biopolymer. It is noticed that one of the important characteristics of fluctuations of this profile is noise flatness [15]. In spite of the obvious significance of noise flatness in ratchets, it seems that analysis of the behavior of SR characteristics depending on noise flatness is still missing in literature. This is because, due to mathematical simplicity, most analysis of SR in systems with colored noise used to be restricted either to cases of Gaussian colored noise

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or to dichotomous noise, while the flatness is constant for both the dichotomous noise and Gaussian colored noise, being equal to 1 and 3, respectively.

Inspired by the fact that the harmonic oscillator is the simplest toy model for different phenomena in nature and as such it is a typical theoretician's paradigm for various fundamental conceptions [16], the authors of Ref. [11] have investigated the long-time behavior of the first two moments and the correlation function for the output signal of a harmonic oscillator with fluctuating frequency subjected to an external periodic force and an additive thermal noise. The colored fluctuations of the oscillator frequency were modeled as a three-level Markovian noise. This linear model of a noisy harmonic oscillator enables exact solutions for moments of the output signal and predicts some unexpected effects in the behavior of SR characteristics. In particular, a multiresonancelike behavior of the variance and the signal-to-noise ratio as function of the noise correlation time are observed and a connection between the occurrence of energetic instability and stochastic multiresonance is established. It is remarkable that for the three-level noise (also called trichotomous noise) used in [11] the flatness parameter can have any value from one to infinity and thus constitutes a case admitting investigations of SR phenomena versus noise flatness.

The main purpose of this paper is to demonstrate, based on the exact expressions of several SR characteristics found in our previous work [11], that stochastic resonance is manifested in the dependence of SR characteristics for the noisy harmonic oscillator [such as spectral amplification (SPA), SNR, and variance of the output signal] upon noise flatness. Furthermore, we will show that in certain parameter regions the SPA and SNR exhibit a resonancelike nonmonotonic behavior versus the values of the friction coefficient. Thus the values of SPA and SNR can be controlled, i.e., either enhanced or suppressed, by changing the friction coefficient. To our knowledge, both the above-mentioned resonance phenomena are new noise-induced effects that have never been observed, let alone discussed before.

As in our previous work [11], we consider the stochastically perturbed harmonic oscillator with a random frequency,

$$\ddot{X} + \gamma\dot{X} + [\omega^2 + Z(t)]X = A_0 \sin \Omega t, \tag{1}$$

where $\dot{X} \equiv dX/dt$, $X(t)$ is the oscillator displacement, and γ is a damping parameter. Fluctuations of the frequency ω^2 are expressed by a Markovian trichotomous noise $Z(t)$, which consists of jumps between three values: $z_1=a$, $z_2=0$, $z_3=-a$, and $a>0$. The jumps follow, in time, the pattern of a Poisson process, the values occurring with the stationary probabilities $p_s(a)=p_s(-a)=q$ and $p_s(0)=1-2q$, where $0<q<1/2$. In a stationary state the fluctuation process $Z(t)$ satisfies

$$\langle Z(t)\rangle = 0, \quad \langle Z(t+\tau)Z(t)\rangle = 2qa^2e^{-\nu\tau}, \tag{2}$$

where the switching rate ν is the reciprocal of the noise correlation time $\tau_c=1/\nu$, i.e., $Z(t)$ is a symmetric zero-mean exponentially correlated noise [17]. The trichotomous process is a particular case of a kangaroo process [13] with the flatness parameter

$$\kappa = \frac{\langle Z^4(t)\rangle}{\langle Z^2(t)\rangle^2} = \frac{1}{2q}. \tag{3}$$

Thus, for the trichotomous noise, flatness is determined by the parameter q, which regulates the relative amount of time spent in the state $z=0$ as opposed to the states $z=a$ and $z=-a$, and as such is a good measure of how close to zero the value of Z stays on the average. We will restrict ourselves to the case where for all states of the trichotomous noise the frequency of the oscillator is positive, i.e.,

$$a < \omega^2. \tag{4}$$

Using the Shapiro-Loginov procedure [18], the exact expressions of the first and second moments of the displacement $X(t)$ as well as for the correlation function

$$K(\tau,t) = \langle X(t+\tau)X(t)\rangle - \langle X(t+\tau)\rangle\langle X(t)\rangle \tag{5}$$

have been calculated in [11]. Particularly, the solution for the first moment $\langle X(t)\rangle$ is stable for all values of the system parameters if inequality (4) holds. In the long-time limit, $t\to\infty$, the moment $\langle X(t)\rangle$ is given by

$$\langle X\rangle_{as} \equiv \langle X\rangle_{t\to\infty} = A \sin(\Omega t + \varphi), \tag{6}$$

where both the output amplitude A and the phase φ of the output average depend on system parameters [see Eqs. (A1)–(A3)]. The necessary and sufficient condition for the stability of second moments (i.e., energetic stability) reads as

$$a^2 < a_{cr}^2 = \frac{\omega^2\gamma(\gamma+\nu)[4\omega^2+\nu(2\gamma+\nu)]^2}{16\omega^2\gamma(\gamma+\nu)+2q\nu[4\omega^2\nu+(2\gamma+\nu)^3]}. \tag{7}$$

In this paper we shall assume that condition (7) is fulfilled. Beside SPA,

$$\mathrm{SPA} := \frac{A^2}{A_0^2}, \tag{8}$$

two another important SR characteristics are the time-homogeneous part of the variance of the oscillator displacement X defined as

$$\sigma^2(X) := \frac{1}{T}\int_0^T (\langle X^2(t)\rangle_{as} - \langle X(t)\rangle_{as}^2)dt, \tag{9}$$

with $T=2\pi/\Omega$ [see also Eqs. (A4)–(A6)], and the output SNR [11,19]. According to Ref. [19], the output SNR R is defined in terms of the Fourier cosine transform of the coherent and incoherent parts of the average of the two-time correlation function at the asymptotic limit $t\to\infty$ over a period T of the external driving, i.e.,

$$R := \frac{\Gamma_1}{\Gamma_2}, \tag{10}$$

where the coherent part Γ_1 and the incoherent part Γ_2 are given by [11,19]

$$\Gamma_1 = \frac{2}{T^2}\int_0^T d\tau \cos(\Omega\tau)\int_0^T \langle X(t)\rangle_{as}\langle X(t+\tau)\rangle_{as}dt = \frac{A^2}{2}, \tag{11}$$

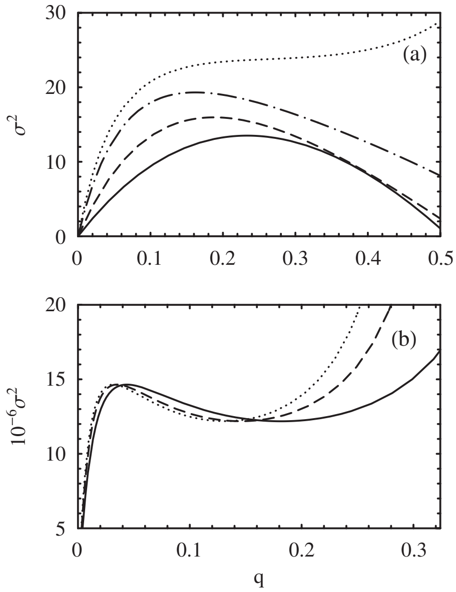

FIG. 1. Variance of the output signal (σ^2) vs the noise parameter q at $A_0=\omega=\Omega=1$. Panel (a), parameter values: $\gamma=0.1$, $a=0.8$; solid line, $\nu=0.01$; dashed line, $\nu=0.05$; dashed dotted line, $\nu=0.15$; dotted line, $\nu=0.25$. Panel (b), σ^2 vs q at different values of the noise amplitude a for $\gamma=10^{-4}$ and $\nu=0.4$. Solid line, $a=0.034$; dashed line, $a=0.038$; dotted line, $a=0.04$. The respective critical values of q, at which energetic instability appears, are $q_{cr1}\approx 0.449$, $q_{cr2}\approx 0.360$, and $q_{cr3}\approx 0.325$.

$$\Gamma_2=\frac{2}{\pi T}\int_0^\infty d\tau\cos(\Omega\tau)\int_0^T[K(\tau,t)]_{as}dt. \qquad (12)$$

Here we emphasize that for all figures throughout this work we use a dimensionless formulation of the dynamics with $\omega=1$ and $A_0=1$.

Next we consider the dependence of several SR characteristics (SPA, σ^2, and R) on the noise flatness $\kappa=1/2q$. The qualitative behavior of SPA, variance σ^2, and output SNR versus κ is sensitive to values of other system parameters. In the case exposed in Fig. 1(a) the variance exhibits a single-peak form of SR at small and moderate values of the noise switching rate ν. As ν increases the SR disappears and in this case the variance is rather an increasing function of q. It is remarkable that in the transition regime [$\nu\approx 0.25$ in Fig. 1(a)] the variance is nearly constant over a finite range of the values of q. The phenomenon of noise-flatness-induced SR for the output variance is not restricted to a simple single-peak form of SR. Figure 1(b) depicts a more complicated behavior of the variance as a function of q for different values of the noise amplitude. In the parameter regimes considered in Fig. 1(b), for increasing values of q the variance starts from zero, increasing to a local maximum, next it decreases, attaining a local minimum, and then σ^2 tends to infinity as q tends to a value $q_{cr}<0.5$. Such a combined SR phenomenon, i.e., first an enhancement, next a suppression, and finally a rapid increase in the output variance is significantly associated with the critical characteristics of stochastic parametric resonance. Namely, the critical value q_{cr} of the noise parameter q at which the variance tends to infinity corresponds to the appearance of noise-induced energetic in-

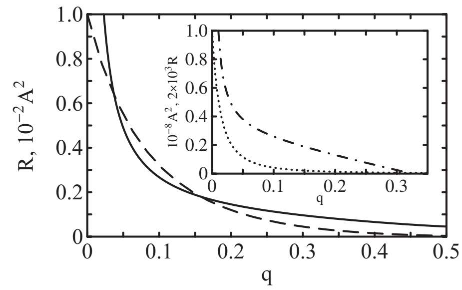

FIG. 2. SNR (R) and SPA (A^2) vs the probability q at $A_0=\omega=\Omega=1$. Solid line (R vs q) and dashed line (A^2 vs q) correspond to the parameters: $\gamma=0.1$, $\nu=0.15$, and $a=0.8$. The inset: dashed-dotted line (R vs q) and dotted line (A^2 vs q) are computed at $\gamma=0.0001$, $\nu=0.4$, and $a=0.04$. Note that the SNR vanishes at the critical value of $q=q_{cr}\approx 0.325$.

stability [cf. Eq. (7)]. Hence, one key factor of the appearance of SR with two local extrema in σ^2 versus κ is the occurrence of energetic instability at some values of the noise flatness κ. As a rule, in the parameter regimes considered in Fig. 1 the SR phenomenon for SPA and for SNR is absent as both SPA and SNR are monotonically decreasing functions of q (cf. Fig. 2).

As mentioned above, there are certain ranges of system parameters for which the behavior of SR characteristics can be qualitatively different. A plot (Fig. 3) of SPA (A^2) versus the noise parameter q for various other system parameters shows a typical resonance with nonmonotonic behavior of the function $A^2(q)$ on q. One can discern two cases. First, if the noise switching rate ν is relatively small, then the SR phenomenon for SPA is exhibited in the form of a suppression of A^2 at some values of q [cf. the curves (1)–(3) in Fig. 3]. Actually, in the case of very small values of the damping parameter γ and the switching rate ν the effect of suppression can be very strong, i.e., at the local minimum of the function $A^2(q)$ the SPA tends to zero. Second, in the case of moderate values of ν a local enhancement of SPA versus q occurs [curve (4) in Fig. 3]. It is remarkable that the peak of $A^2(q)$ depends on ν quite strongly as both its magnitude and its position change. For example, if ν increases, the position of the peak shifts toward greater values of the noise parameter q. In the cases mapped in Fig. 3 with the curves (1)–(3),

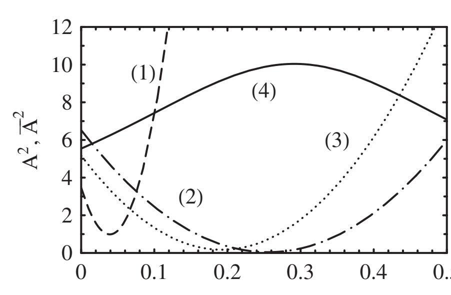

FIG. 3. Dependence of SPA (A^2) on the noise parameter q for $A_0=\omega=1$. (1) Dashed line, $\Omega=0.68$; (2) dashed-dotted line, $\Omega=0.78$; (3) dotted line, $\Omega=1.2$; other parameter values: $\nu=0.01$, $\gamma=0.001$, and $a=0.56$. (4) Solid line $\bar{A}^2(q)=0.2A^2(q)$: $\nu=1.6$, $\gamma=0.001$, $a=0.95$, and $\Omega=0.9$.

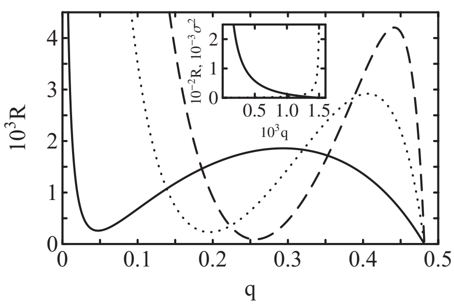

FIG. 4. SNR (R) vs the noise parameter q at $A_0=\omega=1$, $\gamma=0.001$, $a=0.56$, and $\nu=0.01$. Solid line, $\Omega=0.68$; dashed line, $\Omega=0.78$; dotted line, $\Omega=1.2$. The critical value of q is $q_{cr}\approx 0.482$. The inset depicts R (solid line) and σ^2 (dotted line) vs q in the parameter regime $\nu=1.6$, $\gamma=0.001$, $a=0.95$, and $\Omega=0.9$; $q_{cr}\approx 0.00151$.

for the variance σ^2 the SR phenomenon is absent; σ^2 increases monotonically to infinity as q tends to q_{cr} ($q_{cr}<0.5$) where instability appears. Note that SPA is determined with the first-order moment of the displacement of the oscillator and is therefore always stable. In contrast to the variance, the output SNR (R) vs q exhibits SR (see Fig. 4). For increasing values of q the SNR starts from infinity, decreasing to a local minimum, next it increases, attaining a local maximum and then R decreases relatively quickly to zero as q tends to q_{cr}. The behavior of SNR at $q=0$ is a simple consequence of the circumstance that at $q=0$ the oscillator [Eq. (1)] behaves as a deterministic oscillator with $z=0$, and therefore the noise output spectral density Γ_2 [see Eqs. (10) and (12)] tends to zero. At energetic instability, $q=q_{cr}$, the incoherent part of the output correlation function is very large and Γ_2 tends to infinity—thus the SNR tends to zero. It is notable that in accordance with Eqs. (8), (10), and (12) the local minimum of the SNR corresponds to the minimum of the SPA. For an illustrative purpose, at $q=0.25$, some typical realizations of $X(t)$ that correspond to the parameter regime (2) in Fig. 3 are represented in Fig. 5. Note that in this case a desynchronization of the realizations of $X(t)$ appears, which causes a strong suppression of SPA and SNR (cf. Figs. 3 and 4).

Let us note that the SR phenomenon versus noise flatness also appears in the case of adiabatic noise (i.e., in the case of $\nu\to 0$). At a long correlation time, $\nu\to 0$, it follows that by conditions

$$a^2>(\omega^2-\Omega^2)^2-\Omega^2\gamma^2>0, \quad \gamma^2>\Omega^2-2\omega^2, \tag{13}$$

the SPA reaches the minimum

$$\frac{A_{min}^2}{A_0^2}=\frac{4\Omega^2\gamma^2(\omega^2-\Omega^2)^2}{\prod_{n=1}^{3}[(\omega^2-\Omega^2-z_n)^2+\Omega^2\gamma^2]} \tag{14}$$

at

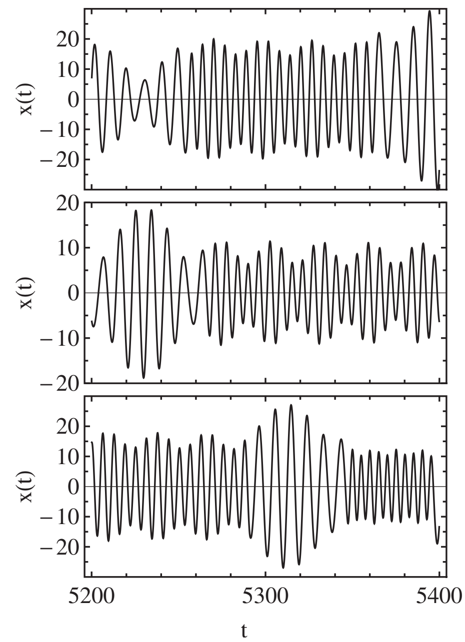

FIG. 5. Three realizations of $X(t)$ at the parameter regime exposed by curve (2) in Fig. 3; $q=0.25$. The mean value of the oscillator displacement $\langle X(t)\rangle$ oscillates between ± 0.21 with the frequency $\Omega=0.78$ (not shown).

$$q=q_m\equiv\frac{1}{2a^2}[a^2+\Omega^2\gamma^2-(\omega^2-\Omega^2)^2]. \tag{15}$$

Note that the inequalities [Eq. (13)] are necessary and sufficient conditions for the SR phenomenon of SPA to occur in the adiabatic limit. Evidently, if the damping parameter γ is low, the suppression of SPA at $q=q_m$ is very pronounced, i.e., A_{min}^2 tends to zero as γ vanishes. In particular, the SR phenomenon of SPA is accompanied by a strong suppression of SNR at $q=q_m$ (cf. Fig. 6). The necessary and sufficient conditions for the existence of resonancelike amplification of the output variance σ^2 read as

$$a^2>(\omega^2-\Omega^2)^2+\Omega^2\gamma^2, \quad \gamma^2<2\omega^2-\Omega^2. \tag{16}$$

It can be shown that the maximum of σ^2 exhibits at

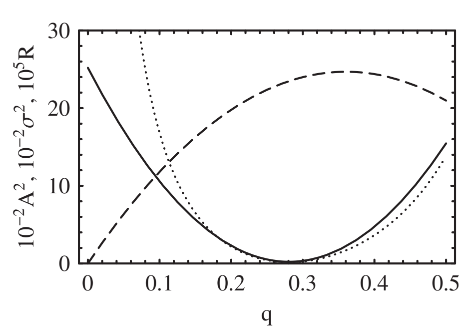

FIG. 6. Dependence of three SR characteristics: output SNR (R), variance (σ^2), and SPA (A^2), on the noise parameter q in the case of a long correlation time. The curves correspond to the following parameters: $A_0=\omega=1$, $a=0,03$, $\Omega=0.99$, $\gamma=0.001$, and $\nu=0.0001$. Solid line, A^2 vs q; dashed line, σ^2 vs q; dotted line, R vs q.

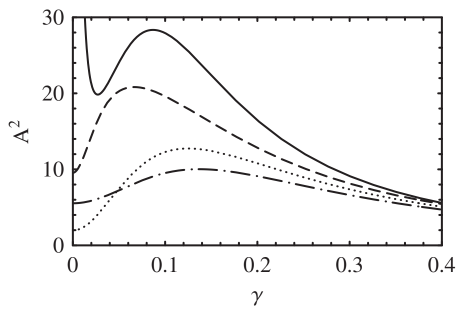

FIG. 7. Spectral amplification (A^2) vs the friction coefficient γ for some values of the noise amplitude a. All quantities are dimensionless, with $A_0=\omega=1$. Dashed line, $a=0.2$; dotted line, $a=0.25$; dashed-dotted line, $a=0.3$; other parameter values: $\nu=0.0001$, $q=0.2$, and $\Omega=0.91$. Solid line, $\nu=0.0001$, $a=0.1$, $q=0.472$, and $\Omega=1$.

$$q = q_{max} = \frac{1}{4a^2}[a^2 + \Omega^2\gamma^2 + (\omega^2 - \Omega^2)^2]. \tag{17}$$

From Eq. (17) it follows that in the adiabatic regime the values of noise flatness κ at which the maximal amplification of the output variance occurs are in the interval (1,2). A comparison of Figs. 3, 4, and 6 shows the vital role of energetic instability at formation of a local maximum of SNR. Contrary to the parameter regimes considered in Figs. 3 and 4, where the energetic instability appears, in the case of Fig. 6 energetic instability is absent for all values of q.

Next we consider the dependence of SPA and SNR on the friction coefficient γ. In Fig. 7 several graphs depict the behavior of the SPA (A^2) versus γ for different representative values of the noise parameters. These graphs show a typical resonancelike behavior of $A^2(\gamma)$. As a rule, the maximal value of A^2 increases as the noise amplitude a decreases, while the positions of the maxima are monotonically shifted to higher γ with a rise in a. Note that the output variance σ^2 is a monotonically decreasing function of γ for all values of the system parameters. Although the dependence of A^2 on the system parameters is generally very complicated [11] and thus simple analytical conditions for the appearance of the resonance behavior of SPA versus γ are not available, it is possible for some particular cases. In particular for the case $q=1/2$ and $\Omega=\omega$ we obtain that the amplitude of the output signal A reaches a maximum vs γ as the conditions

$$a^2 > \frac{2\omega^2\nu^2(\nu^2+4\omega^2)}{\nu^2+2\omega^2}, \quad \nu < \frac{\omega}{2}\sqrt{\sqrt{65}-7} \tag{18}$$

are fulfilled. Thus the effect is possible if the noise amplitude is greater than a threshold value [Eq. (18)], which grows with ν increasing. For small noise correlation times $\tau_c=1/\nu$ system (1) behaves as a deterministic oscillator with an averaged frequency ω and hence the phenomenon of amplification of A vs γ is absent. In the particular case of $\Omega=\omega$, at the long-correlation-time limit $\nu\to 0$ the behavior of SPA as a function of the friction coefficient γ involves two local extrema (cf. the solid line in Fig. 7) in the small interval of the noise flatness, $\kappa\in(1,1.125)$. The positions of the minimum γ_- and of the maximum γ_+ are determined by

$$\gamma_\pm = \frac{a}{\omega}\sqrt{3q-1 \pm \sqrt{q(9q-4)}}. \tag{19}$$

The corresponding extremal values of the SPA read as

$$\frac{A_\pm^2}{A_0^2} = \frac{\{q \pm \sqrt{q(9q-4)}\}^2}{a^2\{3q-1 \pm \sqrt{q(9q-4)}\}\{3q \pm \sqrt{q(9q-4)}\}^2}. \tag{20}$$

Evidently, the maximum of SPA increases rapidly as the noise amplitude a decreases. Note that in accordance with Eq. (18) the threshold value of the noise amplitude tends to zero if $\nu\to 0$. It is important, in view of possible experiments, that at the maximum of SPA the relative variance of the output signal is independent of the noise amplitude and is lower than one:

$$\left(\frac{\sigma^2}{A^2}\right)_{|\gamma=\gamma_+} = \frac{q}{q+\sqrt{q(9q-4)}} < 1. \tag{21}$$

As in the case of the conventional resonance phenomenon (vs Ω) for a classical periodically driven underdamped oscillator, the resonance versus γ of a stochastic oscillator [Eq. (1)] can also be characterized by a phase lag φ between the periodic driving force and the periodic response of the system [see also Eq. (6)] that passes through $\varphi=-\pi/2$ when the friction coefficient γ is tuned through the range where resonance occurs. For example in the case considered above ($\nu=0$, $q=1/2$, and $\Omega\to\omega$) the resonant value of the friction coefficient γ is $\gamma_+=a/\omega$ and the phase lag φ behaves as

$$\tan\varphi = \frac{-\omega\gamma(\omega^2\gamma^2+a^2)}{(\omega^2-\Omega^2)(\omega^2\gamma^2-a^2)}. \tag{22}$$

Thus, γ is increased gradually from zero and swept through the resonant value γ_+, φ decreases from zero, passes through $-\pi/2$ when $\gamma=\gamma_+$, and approaches a minimum when $\gamma=(\sqrt{2+\sqrt{5}})\gamma_+$ (it is assumed that $\Omega>\omega$ when $\Omega\to\omega$).

This simple example also explains why the variation in SPA with γ is nonmonotonic, whereas the corresponding variation in the amplitude of a classical oscillator with γ is well known to be monotonically decreasing when γ increases. The key factor is that realizations of the stochastic oscillator displacement $X(t)$ can be synchronized by increasing the friction coefficient γ. More precisely, consider an ensemble of realizations of the stochastic oscillator for each of which a particular sequence of switching times, between the states of the dichotomous noise $Z(t)$, is chosen from the distribution of switching times. For a given time moment t the relative amount of realizations for the noise states $z_1=a$ and $z_2=-a$ is 1/2. As the switching rate ν is very small, $\nu\to 0$, there is, between two switchings of the noise, enough time that the transition regimes of the oscillator realizations disappear and the average displacement $\langle X\rangle$ can be considered as a sum of the two classical oscillators with the frequencies $\sqrt{\omega^2-a}$ and $\sqrt{\omega^2+a}$. In the case of $\Omega=\omega$, the amplitudes of the output signal are equal, $A_1=A_2=A_0/(2\sqrt{a^2+\omega^2\gamma^2})$ for both oscillators, but the phase lags φ_1 and φ_2 are different: $\varphi_2=-(\pi+\varphi_1)$ and $\sin\varphi_1=-\omega\gamma/\sqrt{a^2+\omega^2\gamma^2}$. The resultant amplitude A of $\langle X\rangle$

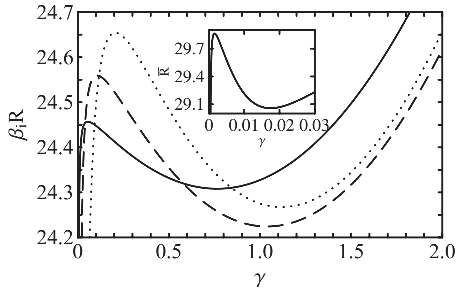

FIG. 8. A plot of the dependence of SNR (R) on the friction coefficient γ for some values of the noise switching rate ν. Parameter values: $A_0=\omega=1$, $q=0.5$, $a=0.4$, and $\Omega=0.1$. Solid line, $\beta_1=27$, $\nu=0.05$; dashed line, $\beta_2=15.5$, $\nu=0.1$; dotted line, $\beta_3=10$, $\nu=0.2$. The inset depicts $\bar{R}=10^4R$ vs γ in the parameter regime $\nu=0.001$, $a=0.75$, and $\Omega=0.51$.

forms as the product of the two terms, namely, $A=2A_1|\sin\varphi_1|$. Contrary to $A_1(\gamma)$, which is the decreasing function of γ and tends to 0 as $\gamma\to\infty$, the factor $|\sin\varphi_1|$ increases monotonically from zero to 1 as γ increases. Hence the maximal value of A forms at an intermediate value of the friction coefficient $\gamma=a/\omega$. We note that at resonance the difference of the phase lags $\varphi_2-\varphi_1$ equals $-\pi/2$.

The phenomenon of γ-induced resonance is not restricted to SPA. Figure 8 depicts the friction-induced resonance for the output SNR at different values of the switching rate ν. In the case of a long correlation time, $\nu=1/\tau_c\ll 1$, this relatively weak effect occurs when the driving frequency Ω is close to the frequencies $\Omega_1\approx\sqrt{\omega^2-a}$, $\Omega_2\approx\omega$, and $\Omega_3\approx\sqrt{\omega^2+a}$, which correspond to the noise states $z_1=-a$, $z_2=0$, and $z_3=a$, respectively. As the switching rate ν is small, the existence of two extrema (i.e., a minimum and a maximum) in the dependence of SNR on γ can be readily inferred by physical intuition. As γ decreases the increase in the incoherent part of the output correlation function is fast enough to suppress R. If γ is sufficiently small, the SNR grows as γ decreases because of the rapid increase in the output amplitude A due to strong resonance of some realizations of $X(t)$ at Ω_i. Particularly, in the noise state z_i all these realizations are strongly synchronized due to the phase lag $\varphi=-\pi/2$ between the periodic driving force and the periodic response of the system by resonance. As γ tends to γ_{cr}, which corresponds to the appearance of energetic instability [see Eq. (7)], i.e., $a_{cr}(\gamma_{cr})=a$, the drastic increase in the output variance involves a rapid decay of the SNR to zero. Finally we emphasize that the results obtained for the friction induced resonance are applicable also in the case of the more conventional dichotomous noise, which is a particular case of trichotomous noise ($q=1/2$).

Our exact analytical results [11] can be a good starting point for more in depth investigations. Here we briefly mention three possible directions concerning the role of noise flatness in the dynamics of a stochastic oscillator. First, it would be interesting to investigate in more detail the phenomena of hypersensitive response to noise amplitude, and for the bona fide resonance, the anomalous strong dependence of the SR gain on the frequency of the driving force at very small intensities of multiplicative noise reported in [11]. Both phenomena occur only at very large values of noise flatness. Second, in the case of additive white noise it is demonstrated that by bona fide SR there exists an exact relation between three common quantifiers for SR, namely, SNR, SPA, and a hysteresis loop area [10]. Thus, it is important to investigate not only SNR, SPA, and variance but also the behavior of other quantifiers of SR, e.g., the hysteresis loop area. Finally, our paper is restricted to the case of trichotomous noise. However, in many physical and biochemical systems, fluctuations have a more general structure, e.g., fluctuations behave as a kangaroo process [13,20,21]. We believe that the model discussed in this paper can be expanded to one that is suitable for studying SR phenomena versus noise flatness in the case of the general kangaroo process.

By conclusion, on the basis of the harmonic oscillator with fluctuating frequency subjected to an external periodic force we have proved the existence of noise-flatness-induced stochastic resonance as well as a friction-generated amplification of the output signal of the oscillator. The advantage of the latter effect is that the control parameter is the friction coefficient (the damping coefficient), which can easily be varied in possible experiments as well as in potential technological applications, e.g., a variable resistor in electric oscillator devices. Stochastically driven harmonic oscillators have been successfully applied in the description of a wide variety of problems in nature [20,22]. For this reason, we believe that the results of this paper not only supply the phenomena to theoretical investigations of SR but also suggest a possibility to design experiments and observations in the field of SR.

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APPENDIX: FORMULAS FOR VARIANCE AND SPA

Using the Shapiro-Loginov procedure [18] by Eq. (1), exact expressions of the first and second moments of the displacement $X(t)$ have been calculated in [11]. To facilitate reading the present article we will repeat (from [11]) the main formulas for SPA and variance that are relevant for this work.

1. Formulas for the first moment

The amplitude A and the phase lag φ in Eq. (6) are given by

$$A^2=A_1^2+A_2^2=\frac{A_0^2[f_1^2+(f_2+2qa^2)^2]}{f_3^2+f_4^2}, \tag{A1}$$

$$\tan\varphi=\frac{f_1f_3-f_4(f_2+2qa^2)}{f_1f_4+f_3(f_2+2qa^2)}, \tag{A2}$$

where

$$f_1 = 2\Omega(\gamma + 2\nu)[\nu(\gamma + \nu) + \omega^2 - \Omega^2],$$

$$f_2 = [\nu(\gamma + \nu) + \omega^2 - \Omega^2]^2 - \Omega^2(\gamma + 2\nu)^2 - a^2,$$

$$f_3 = [f_2(\omega^2 - \Omega^2) - \Omega\gamma f_1] - 2qa^2\nu(\nu + \gamma),$$

$$f_4 = [\Omega\gamma f_2 + (\omega^2 - \Omega^2)f_1] - 4qa^2\nu\Omega. \quad \text{(A3)}$$

2. Formulas for variance

The time-homogeneous part of the variance $\sigma^2(X)$ of the oscillator displacement [see Eq. (9)] can be expressed as

$$\sigma^2(X) = \frac{S_1}{S_2} - \frac{A^2}{2}, \quad \text{(A4)}$$

where

$$S_1 = A_0\{(2\gamma + \nu)\{8A_{11} + 4(2\gamma + \nu)A_9 - [4\omega^2 + \nu(2\gamma + \nu)] \times [2A_7 + (2\gamma + \nu)A_5]\} + (\gamma + \nu)(\gamma A_1 + A_3) \times \{[4\omega^2 + \nu(2\gamma + \nu)]^2 - 16a^2\} + 8qa^2\nu(2A_3 - \nu A_1)\},$$

$$S_2 = 2\omega^2\gamma(\gamma + \nu)[4\omega^2 + \nu(2\gamma + \nu)]^2\left[1 - \left(\frac{a}{a_{cr}}\right)^2\right], \quad \text{(A5)}$$

and a_{cr} is determined by Eq. (7). The coefficients A_1, A_3, A_5, A_7, A_9, and A_{11} are given by the formulas

$$A_1 = \frac{A_0[f_1f_4 + f_3(f_2 + 2qa^2)]}{f_3^2 + f_4^2},$$

$$A_2 = \frac{A_0[f_1f_3 - f_4(f_2 + 2qa^2)]}{f_3^2 + f_4^2},$$

$$A_3 = -\Omega A_2, \quad A_4 = \Omega A_1,$$

$$A_5 = A_0 + (\Omega^2 - \omega^2)A_1 + \Omega\gamma A_2,$$

$$A_6 = (\Omega^2 - \omega^2)A_2 - \Omega\gamma A_1,$$

$$A_7 = \nu A_0 + [\nu(\Omega^2 - \omega^2) + \gamma\Omega^2]A_1 + \Omega[\nu\gamma - (\Omega^2 - \omega^2)]A_2,$$

$$A_8 = \Omega A_0 + \Omega[(\Omega^2 - \omega^2) - \nu\gamma]A_1 + [\gamma\Omega^2 + \nu(\Omega^2 - \omega^2)]A_2,$$

$$A_9 = \Omega A_8 - \omega^2A_5 - (\nu + \gamma)A_7,$$

$$A_{10} = -\Omega A_7 - \omega^2A_6 - (\nu + \gamma)A_8,$$

$$A_{11} = \nu A_9 - \Omega A_{10} - 2qa^2\nu A_1. \quad \text{(A6)}$$

Evidently, if the noise amplitude a tends to the critical value a_{cr}, the variance $\sigma^2(X)$ diverges.

[1] G. Volpe, S. Perrone, J. M. Rubi, and D. Petrov, Phys. Rev. E **77**, 051107 (2008).
[2] P. Reimann, Phys. Rep. **361**, 57 (2002); R. Tammelo, R. Mankin, and D. Martila, Phys. Rev. E **66**, 051101 (2002).
[3] S. Kim, S. H. Park, and C. S. Ryu, Phys. Rev. Lett. **78**, 1616 (1997); A. Sauga and R. Mankin, Phys. Rev. E **71**, 062103 (2005).
[4] R. Kawai, X. Sailer, L. Schimansky-Geier, and C. Van den Broeck, Phys. Rev. E **69**, 051104 (2004); R. Mankin, T. Laas, A. Sauga, A. Ainsaar, and E. Reiter, *ibid.* **74**, 021101 (2006).
[5] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998); T. Wellens, V. Shatokhin, and A. Buchleitner, Rep. Prog. Phys. **67**, 45 (2004).
[6] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981).
[7] V. Berdichevsky and M. Gitterman, Phys. Rev. E **60**, 1494 (1999); M. Gitterman, *ibid.* **67**, 057103 (2003).
[8] Hu Gang, T. Ditzinger, C. Z. Ning, and H. Haken, Phys. Rev. Lett. **71**, 807 (1993).
[9] M. Gitterman, Physica A **352**, 309 (2005).
[10] M. Evstigneev, P. Reimann, C. Schmitt, and C. Bechinger, J. Phys.: Condens. Matter **17**, S3795 (2005).
[11] R. Mankin, K. Laas, T. Laas, and E. Reiter, Phys. Rev. E **78**, 031120 (2008).
[12] V. I. Klyatskin, *Stochastic Equations through the Eye of the Physicist: Basic Concepts, Exact Results, and Asymptotic Approximations* (Elsevier, Amsterdam, 2005).
[13] C. R. Doering, W. Horsthemke, and J. Riordan, Phys. Rev. Lett. **72**, 2984 (1994).
[14] T. C. Elston and C. R. Doering, J. Stat. Phys. **83**, 359 (1996).
[15] M. Bier and R. D. Astumian, Phys. Rev. Lett. **76**, 4277 (1996); R. D. Astumian and M. Bier, Biophys. J. **70**, 637 (1996).
[16] C. C. Bloch, *Classical and Quantum Oscillator* (Wiley, New York, 1997).
[17] R. Mankin, A. Ainsaar, and E. Reiter, Phys. Rev. E **60**, 1374 (1999).
[18] V. E. Shapiro and V. M. Loginov, Physica A **91**, 563 (1978).
[19] J. Casado-Pascual, J. Gómez-Ordóñez, M. Morillo, and P. Hänggi, Phys. Rev. E **68**, 061104 (2003).
[20] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
[21] M. Kostur and J. Luczka, Acta Phys. Pol. B **30**, 27 (1999).
[22] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943); K. Lindenberg, V. Seshadri, and B. J. West, Phys. Rev. A **22**, 2171 (1980); J. I. Jiménez-Aquino, R. M. Velasco, and F. J. Uribe, Phys. Rev. E **77**, 051105 (2008); F. K. Abdullaev, S. A. Darmanyan, and M. R. Djumaev, Phys. Lett. A **141**, 423 (1989); P. S. Landa and A. A. Zaikin, Phys. Rev. E **54**, 3535 (1996); P. S. Landa, A. A. Zaikin, M. G. Rosenblum, and J. Kurths, *ibid.* **56**, 1465 (1997); C.-I. Um, K. H. Yeon, and T. F. George, Phys. Rep. **362**, 63 (2002).